5. FIELD EXTENSIONS

§5.1. Field Extensions as Vector Spaces

Having got the preliminaries out of the way we are now ready to begin Galois Theory proper. A **field extension K/F** is a pair of fields, F ≤ K. We call K an **extension** of F.

Theorem 1: If F ≤ K then K is a vector space over F.

Proof: It's a simple matter to check that the vector space axioms are direct consequences of the field axioms. The elements of F are not only scalars, but being elements of K they are also vectors. The traditional distinction between vectors and scalars that causes us to write vectors differently to scalars must be dispensed with. In fact there's nothing in the definition of a vector space that says that scalars can't also be vectors.

We define the **degree of an extension** K/F to be the dimension of K as a vector space over F and denote it by **|K/F|**. Clearly |F/F| = 1 for any field F.

Examples 1:
(1) |ℂ/ℝ| = 2 since {1, i} is a basis. These 'vectors' span ℂ over ℝ because every complex number can be expressed as a.1 + bi for some a, $b \in$ ℝ.
(2) |ℝ/ℚ| is infinite since a finite-dimensional vector space over ℚ is countable while ℝ is uncountable.

We will only be interested in extensions that are finite-dimensional. An extension K/F is defined to be an **algebraic extension** if K is finite-dimensional over F. So ℂ is an algebraic extension of ℝ while ℝ is not an algebraic extension of ℚ. An extension that is not algebraic is called **transcendental**.

Recall, from an earlier chapter, that we defined algebraic and transcendental numbers (over a field) in terms of being, or not being, a zero of some non-zero polynomial. This new definition applies to extensions and is expressed in terms of dimensions. Are these related? Yes, they are because, as we shall see, α is algebraic over F if and only if F[α] is an algebraic extension of F.

Recall that the intersection of any collection of fields is itself a field. If F is a number field (subfield of ℂ) and $\alpha \in \mathbb{C}$ we define **F[α]** to be the intersection of all fields that contain both F and α. Clearly this is the smallest field that contains F and α. An extension F[α]/F is called a **simple extension**.

Example 2: $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. We've shown that this is a field and clearly any field that contains $\mathbb{Q}$ and $\sqrt{2}$ must contain every number of the form $a + b\sqrt{2}$.

If $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ then $\mathbf{F[\alpha_1, \alpha_2, ..., \alpha_n]}$ is defined to be the intersection of all fields that contain F and also each of the α_i. Even if $n > 1$ this may turn out to be a simple extension.

Example 2: $\mathbb{Q}[\sqrt{2}, i] = \mathbb{Q}[\sqrt{2} + i]$ and so is a simple extension of $\mathbb{Q}$.
Proof: Clearly $\mathbb{Q}[\sqrt{2} + i] \le \mathbb{Q}[\sqrt{2}, i]$. To show the converse we must show that $\mathbb{Q}[\sqrt{2} + i]$ contains both $\sqrt{2}$ and i individually. In other words, if $\alpha = \sqrt{2} + i$ we must show that $\sqrt{2}$ and i can each be expressed in terms of α (using addition, subtraction, multiplication and division).
Now $\alpha^2 = 2 - 1 + 2\sqrt{2}i$
$= 1 + 2\sqrt{2}i$ and
$\alpha^3 = (1 + 2\sqrt{2}i)(\sqrt{2} + i)$
$= \sqrt{2} + 4i + i - 2\sqrt{2}$
$= 5i - \sqrt{2}$.
Hence $\alpha^3 + \alpha = 6i$.

So $i = \dfrac{\alpha^3 + \alpha}{6} \in \mathbb{Q}[\alpha]$ and $\sqrt{2} = \alpha - i \in \mathbb{Q}[\alpha]$.

If $f(x) \in F[x]$ we define **F[a(*x*)]**. to be F extended by all the zeros of a(x) in $\mathbb{C}$. Such an extension is called a **polynomial extension** of F.

Often we write it as **F[a(*x*) = 0]**, or we replace a(x) = 0 by an equivalent equation.

A special case is $F[x^n - \alpha]$ where $\alpha \in F$. We can write it as $F[x^n - \alpha = 0]$, or more usually, $\mathbf{F[x^n = \alpha]}$. We call such an extension a **radical extension** of F.

Example 3: Since the zeros of $x^2 + x + 1$ are ω and ω^2 we have:

$$\mathbb{Q}[x^2 + x + 1] = \mathbb{Q}[\omega, \omega^2] = \mathbb{Q}[\omega].$$

This is both a simple extension and a polynomial extension of $\mathbb{Q}$. In fact it is also a radical extension because it's $\mathbb{Q}[x^3 = 1]$.

Theorem 2: If $F \leq H$ and H is algebraic over F then there exists a polynomial extension K of F such that $F \leq H \leq K$.

Proof: Suppose that $H = F[\alpha_1, ..., \alpha_m]$ and for each i let $p_i(x)$ be the minimum polynomial of α_i over F.

Take $K = F[p_1(x)p_2(x)...p_m(x)]$.

Theorem 3: For all $q \in \mathbb{Q}$, $\sin(2\pi q)$ and $\cos(2\pi q)$ are algebraic over $\mathbb{Q}$ and if $\cos(2\pi q) \neq 0$, $\tan(2\pi q)$ is algebraic over $\mathbb{Q}$.

Proof: Let $q = \frac{m}{n}$ and let $s = \sin(2\pi q)$ and $c = \cos(2\pi q)$.

We are not insisting that m, n are coprime so we may assume that n is even.

Then $(c + \mathrm{i}s)^n = \cos(2\pi m) + \mathrm{i}\sin(2\pi m) = 1$.

Equating imaginary parts we get:

$$\binom{n}{1} c^{n-1}s - \binom{n}{3} c^{n-3}s^3 + \ldots = 0.$$

If $c \neq 0$ we may divide by c^n to get $\binom{n}{1}$ t $- \binom{n}{3} t^3 + \ldots = 0$.

So $\tan(2\pi q)$ is algebraic.

If $s = 0$ then $c = \pm 1$ and the theorem holds. So we may assume that $s \neq 0$.

Dividing the above equation by s we get:

$$\binom{n}{1} c^{n-1} - \binom{n}{3} c^{n-3}s^2 + \ldots = 0.$$

The left hand side only involves even powers of s and so replacing s^2 by $1 - c^2$ we get a suitable polynomial expression in c that is equal to 0. So $\cos(2\pi q)$ is algebraic.

Equating real parts in the above we get:

$$c^n - \binom{n}{2} c^{n-2}s^2 + \binom{n}{4} c^{n-4}s^4 - \ldots = 1.$$

Since n is even we may remove the powers of c by replacing c^2 by $1 - s^2$. So $\sin(2\pi q)$ is algebraic.

§5.2. The Degree of a Simple Extension

The next theorem gives the explicit form of the field F[α] when α is algebraic over F. It says that the elements of F[α] consist of all polynomials over F evaluated at $x = \alpha$.

Theorem 4: If α is algebraic over F and the minimum polynomial, p(x), of α over F has degree n then:

$$F[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 + ... + a_{n-1}\alpha^{n-1} \mid \text{each } a_i \in F\}.$$

Proof:

Let $K = \{a_0 + a_1\alpha + a_2\alpha^2 + ... + a_{n-1}\alpha^{n-1} \mid \text{each } a_i \in F\}$. Clearly K is closed under addition and subtraction. Since α^n can be expressed in terms of lower powers, all powers of α can be expressed in terms of these powers. So K is closed under multiplication. It follows that if k(x) is any polynomial in F[x] then k(α) ∈ K.

To show that K is a field we merely need to show that it is closed under multiplicative inverses. We can express every element of K as f(α) where f(x) ∈ F[x] has degree at most n. Suppose f(α) ≠ 0.

Then p(x) doesn't divide f(x). Since p(x) is prime this means that f(x) and p(x) are coprime. It follows that for some h(x), k(x) ∈ F[x] we have p(x)h(x) + f(x)k(x) = 1.

Hence 1 = p(α)h(α) + f(α)k(α) = f(α)k(α) and so

$$f(\alpha)^{-1} = k(\alpha) \in K.$$

We've shown that K is a field that contains F and α. Hence $F[\alpha] \le K$. But clearly $K \le F[\alpha]$.

Corollary1: If α is algebraic over F then $|F[\alpha]/F|$ is the degree of the minimum polynomial of α over F.

Corollary 2: $F[\alpha]$ is algebraic over F if and only if α is algebraic over F.

Proof: Clearly if α is algebraic over F then $F[\alpha]$ is algebraic (finite-dimensional) over F.

Suppose $|F[\alpha]/F| = n$. Then $1, \alpha, \alpha^2, \ldots, \alpha^n$ are linearly dependent over F, that is $a_0 + a_1\alpha + \ldots + a_n\alpha^n = 0$ for some $a_i \in F$, not all zero. But this means that α is algebraic over F.

Example 4: $\mathbb{Q}[\sqrt[3]{2}\,] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$ since the minimum polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$.

§5.3. Dimensions of Field Extensions

Theorem 5: Suppose V is a finite-dimensional vector space over the field K, which in turn is a finite-dimensional extension of the field F. Then V can be viewed as a vector space over F and:

$$|V/F| = |V/K| \times |K/F|.$$

Proof: Let $|V/K| = n$ and $|K/F| = m$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a basis for V as a vector space over K and let $\beta_1, \beta_2, \ldots, \beta_m$ be a basis for K as a vector space over F. I'll show that the mn products $\alpha_i\beta_j$ form a basis for V as a vector space over F. The theorem then follows.

The $\alpha_i\beta_j$ span V over F.

Let $v \in V$.

Then $v = k_1\alpha_1 + ... + k_n\alpha_n$ for some 'scalars' in K.

But each of these is a 'vector' in K, regarded as a vector space over F. Hence each k_i can be expressed as:

$$k_i = h_{i1}\beta_1 + ... + h_{im}\beta_m$$

where each $h_{ij} \in F$. Substituting into the previous equation we obtain $\sum_{i,j} h_{ij}\alpha_i\beta_j$ showing that the $\alpha_i\beta_j$ span V over F.

The $\alpha_i\beta_j$ are linearly independent over F.

Suppose $\sum_{i,j} h_{ij}\alpha_i\beta_j = 0$ for h_{ij}'s $\in F$.

Let $\delta_i = \sum_j h_{ij}\beta_j$ for each i.

Then $\sum_i \delta_i\alpha_i = 0$.

Since the α_i are linearly independent and each $\delta_i \in K$, each $\delta_i = 0$.

So, for each i, $\sum_j h_{ij}\beta_j = 0$.

Since the β_j are linearly independent and each $h_{ij} \in F$, each $h_{ij} = 0$.

Thus the set of mn products $\alpha_i\beta_j$ is linearly independent, and is hence a basis for V over F.

Example 5: $|\mathbb{Q}[\sqrt[3]{2}]/\mathbb{Q}| = 3$.
$\mathbb{Q}[x^3 = 2] = \mathbb{Q}[\sqrt[3]{2}][\omega]$.
The minimum polynomial of ω over $\mathbb{Q}[\sqrt[3]{2}]$ is $x^2 + x + 1$. This is because $\omega^2 + \omega + 1 = 0$ and because $x^2 + x + 1$ is prime over $\mathbb{Q}[\sqrt[3]{2}]$ (if it was composite ω would be real).
Hence $|\mathbb{Q}[x^3 - 2]/\mathbb{Q}[\sqrt[3]{2}]| = 2$ and so:

$$|\mathbb{Q}[x^3 = 2]/\mathbb{Q}| = 3.2 = 6.$$

Example 6: Is $\sqrt{2} \in \mathbb{Q}[\sqrt[3]{2}]$?
Answer: No. If it was the case then $\mathbb{Q}[\sqrt{2}] \le \mathbb{Q}[\sqrt[3]{2}]$.
But this would mean that $|\mathbb{Q}[\sqrt[3]{2}]/\mathbb{Q}[\sqrt{2}]| = 3/2$ which is impossible.

If F is a number field we define $\mathbf{F}^*$ to be the set of all complex numbers that are algebraic over F.

Theorem 6: If F is a field then so is F^*.
Proof: Let $\alpha, \beta \in F^*$ and
let $|F[\alpha]/F| = m$ and $|F[\beta]/F| = n$.
The degree of the minimum polynomial of β over F will have degree n.
This will be a polynomial over $F[\alpha]$, but it may factorise over this larger field. But certainly the minimum

polynomial of β over $F[\alpha]$ will have degree at most n. Hence $|F[\alpha][\beta]/F[\alpha]| \le n$.

It follows that $|F[\alpha, \beta]/F| \le mn$.

Now $F[\alpha + \beta]$, $F[\alpha\beta]$, $F[-\alpha]$ and, if $\alpha \neq 0$, $F[\alpha^{-1}]$ are all subfields of $[\alpha, \beta]$ and hence are finite-dimensional over F. This shows that F^* is a field.

EXERCISES FOR CHAPTER 5

Exercise 1: For each of the following determine whether it is true or false. Give reasons.

(1) $\mathbb{Q}[\sqrt{8}] = \mathbb{Q}[\sqrt{2}]$.

(2) If F is a number field then so is $\{f(\alpha) \mid f(x) \in F[x]\}$.

(3) $1, \omega, \omega^2$ are linearly independent over $\mathbb{Q}$.

(4) $\mathbb{Q}[1 + \sqrt[3]{2}\,] = \mathbb{Q}[\sqrt[3]{4}\,]$.

(5) If H, K are number fields then $H \cap K$ a number field.

(6) $\{a + b\sqrt[3]{2} \mid a, b \in \mathbb{Q}\}$ is a vector space over $\mathbb{Q}$.

(7) $|\,\mathbb{Q}[e^{2\pi i/11}]/\mathbb{Q}| = 11$.

Exercise 2: Prove that all the zeros of $x^8 + x^4 + 1$ are in $\mathbb{Q}[i][\sqrt{3}]$.

Exercise 3: Prove that

$$\{a + b\sqrt[4]{2} + c\sqrt{2} + d\sqrt[4]{8} \mid a, b, c, d \in \mathbb{Q}\}$$

is a number field. [Use Theorem 7]

Exercise 4: Prove that $\sqrt{2} \in \mathbb{Q}[i][e^{\pi i/4}]$.

Exercise 5: Which of the following are quadratic extensions of $\mathbb{Q}[\sqrt{3}\,]$ and which are $\mathbb{Q}[\sqrt{3}\,]$ itself?

(a) $\mathbb{Q}[\sqrt{3}\,]\,[\sqrt{12}\,]$;

(b) $\mathbb{Q}[\sqrt{3}\,][\sqrt{-5 + 7\sqrt{3}}\,]$;

(c) $\mathbb{Q}[\sqrt{3}\,][\sqrt{7 + 4\sqrt{3}}\,]$;

(d) $\mathbb{Q}[\sqrt{3}\,][\sqrt{1 + 2\sqrt{3}}\,]$.

Exercise 6: Prove that $x^4 - 4x^2 + 2$ is prime over $\mathbb{Q}$. Hence or otherwise show that it is the minimum polynomial of $\sqrt{2 + \sqrt{2}}$.

Exercise 7: Prove that $\sqrt{3 + 2\sqrt{2}} \in \mathbb{Q}[\sqrt{2}\,]$.

Exercise 8: Prove that $\sqrt{-1 + 2\sqrt{2}} \notin \mathbb{Q}[\sqrt{2}\,]$.

Exercise 9: Find the minimum polynomials over $\mathbb{Q}$ of:

(a) $\sqrt{3 + 2\sqrt{2}}$,

(b) $\sqrt{-1 + 2\sqrt{2}}$.

Exercise 10: Find a basis for $\mathbb{Q}[\sqrt{-1 + 2\sqrt{2}}\,]$ as a vector space over $\mathbb{Q}$.

Exercise 11: Find $|\,\mathbb{Q}[2^{1/4} + \sqrt{8}]/\mathbb{Q}|$.

Exercise 12: Prove that there is no number field F such that $\mathbb{R} < F < \mathbb{C}$.

SOLUTIONS FOR CHAPTER 5

Exercise 1: (1) **TRUE**; (2) **FALSE** Only true if α is algebraic; (3) **FALSE** as $1 + \omega + \omega^2 = 0$; (4) **TRUE**; (5) **TRUE**; (6) **TRUE**; (7) **FALSE** The minimum polynomial has degree ≤ 10 since $x - 1$ is a factor of
$x^{11} - 1$.

Exercise 2: $x^8 + x^4 + 1$ is a quadratic in x^4.
Using the quadratic formula $x^4 = \dfrac{-1 \pm \sqrt{3}\,\mathrm{i}}{2} = e^{2\pi/3}$ or $e^{4\pi/3}$.
Hence $x = \pm\, e^{2\pi/3}, \pm\, e^{2\pi/3}\,\mathrm{i}, e^{4\pi/3}, e^{4\pi/3}\,\mathrm{i}$.
Since $e^{2\pi/3}$, $e^{4\pi/3}$ and $\mathrm{i} \in \mathbb{Q}[\mathrm{i}][\sqrt{3}]$ all 8 zeros of
$x^8 + x^4 + 1$ belong to this field.

Exercise 3: The minimum polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$ is
$x^4 - 2$ and hence for every
$f(x) \in \mathbb{Q}[x]$, $f(\sqrt[4]{2}) = a + b\sqrt[4]{2} + c\sqrt{2} + d\sqrt[4]{8}$ for some $a, b, c, d \in \mathbb{Q}$. It follows that

$$\{a + b\sqrt[4]{2} + c\sqrt{2} + d\sqrt[4]{8} \mid a, b, c, d \in \mathbb{Q}\}$$
$$= \{f(\sqrt[4]{2}) \mid f(x) \in \mathbb{Q}[x]\}$$

and, by Theorem 7 this is $\mathbb{Q}[\sqrt[4]{2}]$ which, by definition is a number field.

Exercise 4: $e^{\pi \mathrm{i}/4} = \dfrac{1}{\sqrt{2}} + \dfrac{\mathrm{i}}{\sqrt{2}}$.

Hence $\sqrt{2} = \dfrac{1 + i}{e^{\pi i/4}} \in \mathbb{Q}[i][e^{\pi i/4}]$.

Exercise 5: In other words, which of the following are squares of an element in $\mathbb{Q}[\sqrt{3}]$:
$12, -5 + 7\sqrt{3}, 7 + 4\sqrt{3}, 1 + 2\sqrt{3}$?.
(a) $\sqrt{12} = 2\sqrt{3}$ so $\mathbb{Q}[\sqrt{3}\,][\sqrt{12}\,] = \mathbb{Q}[\sqrt{3}\,]$.
(b) Suppose $-5 + 7\sqrt{3} = (a + b\sqrt{3})^2$ for $a, b \in \mathbb{Q}$.
Then $a^2 + 3b^2 = -5$, which has no solution.
Hence $\mathbb{Q}[\sqrt{3}\,][\sqrt{-5 + 7\sqrt{3}}\,]$ is a quadratic extension of $\mathbb{Q}[\sqrt{3}\,]$.
(c) Suppose $7 + 4\sqrt{3} = (a + b\sqrt{3})^2$ for $a, b \in \mathbb{Q}$.
Then $a^2 + 3b^2 = 7$ and $2ab = 4$.
Hence $b = \dfrac{2}{a}$ and so $a^2 + \dfrac{12}{a^2} = 7$.
So $a^4 - 7a^2 + 12 = 0$
Hence $a^2 = \dfrac{7 \pm \sqrt{49 - 48}}{2} = 3, 4$.
Since $a \in \mathbb{Q}$, $a = \pm 2$ and so $b = \pm 1$.
Thus $(2 + \sqrt{3})^2 = 7 + 4\sqrt{3}$ and so

$$\mathbb{Q}[\sqrt{3}\,][\sqrt{7 + 4\sqrt{3}}\,] = \mathbb{Q}[\sqrt{3}\,].$$

Hence $\mathbb{Q}[\sqrt{3}\,][\sqrt{-5 + 7\sqrt{3}}\,]$ is a quadratic extension of $\mathbb{Q}[\sqrt{3}\,]$.

(d) Suppose $1 + 2\sqrt{3} = (a + b\sqrt{3})^2$ for $a, b \in \mathbb{Q}$.
Then $a^2 + 3b^2 = 1$ and $2ab = 2$.

Thus $b = \dfrac{1}{a}$ and so $a^2 + \dfrac{3}{a^2} = 1$.

So $a^4 - a^2 + 3 = 0$

Therefore $a^2 = \dfrac{7 \pm \sqrt{1 - 12}}{2} \notin \mathbb{R}$.

Hence there is no solution and so $\mathbb{Q}[\sqrt{3}\,][\sqrt{1 + 2\sqrt{3}}\,]$ is a quadratic extension of $\mathbb{Q}[\sqrt{3}\,]$.

Exercise 6: If $\alpha = \sqrt{2 + \sqrt{2}}$ then $\alpha^2 = 2 + \sqrt{2}$ and so $(\alpha^2 - 2)^2 = 2$, which gives

$$\alpha^4 - 2\alpha^2 + 2 = 0.$$

The difficulty is in showing that $x^4 - 4x^2 + 2$ is prime.
The zeros are $\pm\sqrt{2 \pm \sqrt{2}}$ and if any of them was rational, $\sqrt{2}$ would be rational, a contradiction.

But having no rational zeros isn't enough. We must also show that this polynomial doesn't factorise into two quadratics. We could do this by taking the zeros in pairs and showing that the corresponding quadratic does not have rational coefficients. Or we could try to use one of the other methods.

Eisenstein is no use. Nor is there any joy in looking at it modulo 2.

Modulo 3 it becomes $x^4 + 2x^2 + 2$.
The monic prime quadratics over $\mathbb{Z}_3$ are $x^2 + 1$, $x^2 + x + 2$ and $x^2 + 2x + 2$.

We can then check that no product of these (including the squares) is equal to $x^4 + 2x^2 + 2$.

This is the case, so $x^4 - 4x^2 + 2$ is prime over $\mathbb{Q}$ and so must be the minimum polynomial of $\sqrt{2 + \sqrt{2}}$.

Alternatively we could use the Too Many Primes Test. For $x = 0, \pm1, \pm2, \pm3, \pm7$ we get the values 2, –1, 2, 47 and 2207, all of which are prime.

So $x^4 - 4x^2 + 2$ is prime or ± 1 for 9 integer values of x and hence must be prime over $\mathbb{Q}$.

Exercise 7:

Suppose that $\sqrt{3 + 2\sqrt{2}} = a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$.
Then $3 + 2\sqrt{2} = a^2 + 2b^2 + 2ab\sqrt{2}$.
Hence $a^2 + 2b^2 = 3$ and $ab = 1$.
Thus $a^2 + 2\left(\frac{1}{a}\right)^2 = 3$ and so $a^4 - 3a^2 + 2 = 0$.
Therefore $a^2 = \frac{3 \pm \sqrt{9 - 8}}{2} = 2$ or 1.
Clearly $a^2 = 2$ is impossible.
Hence $a = b = 1$ or $a = b = -1$.
So $\sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}$.

Exercise 8: Suppose $\sqrt{-1 + 2\sqrt{2}} = a + b\sqrt{2}$
where $a, b \in \mathbb{Q}$.
Then $-1 + 2\sqrt{2} = a^2 + 2b^2 + 2ab\sqrt{2}$.
Hence $a^2 + 2b^2 = -1$, which is impossible.

Exercise 9:

(a) By Exercise 7, $\sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}$ and so the minimum polynomial is $(x - 1)^2 = 2$, that is, $x^2 - 2x - 1$.

(b) If $x = \sqrt{-1 + 2\sqrt{2}}$ then $(x^2 + 1)^2 = 8$, that is $x^4 + 2x^2 - 7 = 0$.

Having shown that $\sqrt{-1 + 2\sqrt{2}} \notin \mathbb{Q}[\sqrt{2}\,]$ in Exercise 8 we conclude that $|\mathbb{Q}[\sqrt{-1 + 2\sqrt{2}}\,]/\mathbb{Z}\,\mathbb{Q}| = 4$.

Hence the degree of the minimum polynomial of $\sqrt{-1 + 2\sqrt{2}}$ over $\mathbb{Q}$ is 4 and so $x^4 + 2x^2 - 7$ must be its minimum polynomial.

Exercise 10: $|\mathbb{Q}[\sqrt{-1 + 2\sqrt{2}}\,]/\mathbb{Q}| = 4$ so the dimension of $\mathbb{Q}[\sqrt{-1 + 2\sqrt{2}}\,]$ over $\mathbb{Q}$ is 4.

Hence a basis is: 1, $\sqrt{-1 + 2\sqrt{2}}$, $-1 + 2\sqrt{2}$ and $(\sqrt{-1 + 2\sqrt{2}}\,)^3$.

Exercise 11: Note that $\sqrt{8} = 2\sqrt{2} \in \mathbb{Q}[2^{1/2}] \le \mathbb{Q}[2^{1/4}]$.

Hence $\mathbb{Q}[2^{1/4} + \sqrt{8}] = \mathbb{Q}[2^{1/4}]$.

Its degree over $\mathbb{Q}$ is clearly 4.

Exercise 12: $|\mathbb{C}:\mathbb{R}| = 2$ and so, if there was such a field, $|\mathbb{C}:F|.|F:\mathbb{R}| = 2$. One of these factors must be 1 and so either $F = \mathbb{C}$ or $F = \mathbb{R}$.